# A Random Integration Algorithm for High-dimensional Function Spaces

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**Abstract**

We introduce a novel random integration algorithm that enjoys high convergence order for functions characterized by sparse frequencies or rapidly decaying Fourier coefficients. Specifically, for integration in periodic isotropic Sobolev space and the isotropic Sobolev space with compact support, our approach attains a nearly optimal root mean square error (RMSE) bound. In contrast to previous nearly optimal algorithms, our method exhibits polynomial tractability. Our integration algorithm also enjoys nearly optimal bound for weighted Sobolev space. By incorporating the trick of change of variable, our algorithm is proven to achieve the semi-exponential convergence order for the integration of analytic functions, which marks a significant improvement over the previously obtained super-polynomial convergence order. For the integration of Wiener-type functions, the sample complexity of our algorithm (with a slight variant) is independent of the decay rate of Fourier coefficients.

**Keywords:** Numerical integration; Monte Carlo; Curse of dimensionality; Sample complexity; Information-Based complexity
**MSC 2020:** 65D30, 68Q25, 65C05, 42B10

## 1 Introduction

This paper is concerned with numerically integrating multivariate functions defined over the $d$-dimensional unit cube. Denote

$$\mathrm{INT}(f) := \int_{[0,1]^d} f(x)dx.$$

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Evaluating $\mathrm{INT}(f)$ amounts to estimating the value of $\hat{f}(0)$, wherein the Fourier transform is defined as

$$\hat{f}(w) = \int_{[0,1]^d} f(x) \exp(-2\pi i w \cdot x) dx. \tag{1}$$

For a given positive integer $N$, we introduce the following discrete Fourier transform

$$\widetilde{f}^{(N)}(w) = \frac{1}{N^d} \sum_{z \in \mathbb{Z}_N^d} f(z/N) \exp(-2\pi i w \cdot z/N), \quad w \in \mathbb{Z}_N^d \tag{2}$$

where $\mathbb{Z}_N^d := \mathbb{Z}^d \cap [0, N)^d$.

Inspired by the sparse Fourier transform (SFT) from signal processing [2, 15, 23], our approach to estimating $\hat{f}(0)$ involves two key steps. First, we create a hash map that effectively disperses the frequencies, ensuring that the energy of frequencies near zero (excluding zero itself) is small. Second, we employ a low-pass filter to extract the low-frequency components. Here, we utilize the hash mapping $A_N$ developed in our previous paper [2],

$$A_N := \begin{pmatrix} \mathbf{I}_{d-1} & \mathbf{0} \\ \boldsymbol{v} & h_d \end{pmatrix}, \tag{3}$$

where $N$ is some prime number, $\mathbf{I}_{d-1}$ represents the identity matrix of order $d-1$, $\boldsymbol{v} := (h_1, h_2, \ldots, h_{d-1})$, and $(\boldsymbol{v}, h_d)$ is drawn from the uniform distribution on the set $\mathbb{Z}^d \cap [1, N)^d$. Note that the distribution here differs from the setting in [2]. Subsequently, we construct the following low-pass filter:

$$f_{A_N,L,r}(x) := \sum_{|l| \le L} f(\{A_N^\top (x - y_l)\}) G_{r,l}, \tag{4}$$

where $\{t\} \in [0, 1)$ denotes the fractional part of $t$ (for instance, $\{3.2\} = 0.2$, $\{-1.3\} = 0.7$), $y_l = (0, \ldots, 0, l/N)^\top$, and

$$G_{r,l} = \frac{1}{r\sqrt{2\pi}} \exp(-\frac{l^2}{2r^2}).$$

The comprehensive attributes of $G_{r,l}$ are elaborated in Lemma 2.3. By letting $z = A_N^\top x$ and substituting it into Eq.(4), we derive the following critical integration formula

$$I(f)_{H_N,L,r,z} := \sum_{|l| \le L} f\left(\left\{\frac{z - lH_N}{N}\right\}\right) G_{r,l}, \tag{5}$$

where $H_N$ and $z$ are drawn from the uniform distribution over $\mathbb{Z}^d \cap [1, N)^d$ and $\mathbb{Z}^d \cap [0, N)^d$, respectively.

When employing other recent high-dimensional SFT methods for estimating numerical integration, as cited in references [26, 28, 22], it is not feasible to achieve the desired error bounds for functions in Sobolev spaces. Previously, Gilbert et.al [15] introduced an algorithmic framework for high-dimensional SFT, which could potentially be adapted to develop integration algorithms that attain the desired error bound. However, such a

framework would increase the upper bound by a factor of $d^5$, as analyzed on page 61 of [28]. In addition, Eq. (5) can also be viewed as a special random lattice rule. Recently, a series of studies [31, 10, 33, 39, 18] have explored the use of random lattice rules for estimating numerical integration in weighted Sobolev spaces, drawing inspiration from the pioneering contributions of Bakhvalov [1].

Utilizing Eq. (5), along with the local Monte Carlo sampling (see Eq.(12)) and the median trick [27, 35, 32, 20, 40, 41, 21], we shall construct a novel randomized integration algorithm. Our method introduces a fresh understanding in the construction of numerical integration algorithms from the perspectives of filtering and dispersing sparse frequencies. The algorithm possesses several theoretical advantages compared to existing algorithms. We summarize these advantages as follows:

- *Our algorithm is nearly optimal for integration in the periodic isotropic Sobolev space as well as in the isotropic Sobolev space with compact support, while maintaining polynomial tractability. Consequently, it is also nearly optimal and polynomially tractable for smooth functions that vanish on the boundary. In contrast, previous works [30, 43] have developed optimal integration algorithms for the mixed Sobolev space and the isotropic Sobolev space with compact support. However, these algorithms suffer from a drawback: their RMSE upper bounds are not less than* $2^{\Theta(d)}M^{s/d+1/2}$ *(where* $d$*,* $s$*,* $M$ *denotes the dimension, the order and the sample size, respectively, see Remark 3.2 in [43]. In [30], the corresponding constant is larger than* $1/(d^{d/2}(2^{1/d}-1)^d)$ *and the order of the Sobolev space is required to be greater than* $d/2$*, as observed from the proofs of Theorem 1 and Lemma 5 in [30]), rendering them not polynomially tractable in terms of sample complexity. Furthemore, our upper bound for integration in the isotropic Sobolev space with compact support is superior to that presented in [30, 43] when* $s \le \Theta(d)$ *and* $M \le 2^{2^{\Theta(d)}}$ *(namely,* $\log^{\Theta(1)} M \le 2^{\Theta(d)}$*). Recently, [6] proposes a nearly optimal integration algorithm for general smooth functions and smooth functions that vanish on the boundary of* $[0,1]^d$ *(see Theorem 3.2 in [6]). However, as acknowledged in the "Future work" section of [6], they still struggle to overcome the curse of dimensionality in terms of sample complexity. For more details see Theorem 2.7, Remark 2.8, Corollary 2.9 and Remark 2.10 in section 2.*

- *By incorporating the trick of change of variable, the numerical integration of the analytic functions over* $[0,1]^d$ *is proven to achieve semi-exponential convergence order, which marks a significant improvement over the previously obtained superpolynomial convergence order [40, 41]. For more details see Theorem 3.1 and Remark 3.2 in section 3.*

- *Our algorithm (with a slight variant) achieves a convergence rate of order* $\frac{\log(M)}{M}$ *(where* $M$ *denotes the sample size) for integration in the subspace of Wiener algebra. In comparison to previous studies[8, 12, 16, 17, 3, 29, 11], the sample complexity of our algorithm is independent of the decay rate of Fourier coefficients and the Hölder continuity of the functions, provided that we choose a sufficiently large* $N$ *in Eq. (5).* For more details see see Theorem 3.4 and Remarks 3.5 and 3.6. in section 3.

- *Our algorithm is also nearly optimal for integration in weighted Sobolev space. Unlike the algorithms [31, 10, 33, 39, 18], our algorithm does not require prior information about weights, making it universally applicable to varying weights in the RMSE sense. The integration algorithms applicable to different weights under the worst-case scenario have been established in [14, 7, 9, 13, 20]. For more details see Theorem 4.1 and Remark 4.2 in section 4. Very recently, [19] has presented a universal (respect to both weights and order) nearly optimal algorithm for weighted Sobolev spaces in the scenario where the order $s > 1/2$, whereas our nearly optimal algorithm is valid for the case where $s > 0$. These two works are independent of each other and different in terms of algorithms, the earliest version (for the "weighted Korobov space" of order $s > 0$) of our paper has already been uploaded to arXiv [5] (despite containing numerous minor errors, they are easily corrected). For more details see Remark 4.3.*

# 2 Integration in Isotropic Sobolev Spaces

For a function $f$, we consider the random integration algorithms with the random variable $\omega$ of the following form

$$A_{n,w}(f) = \phi_{n,w}(f(x_1), f(x_2), \ldots, f(x_n)),$$

where $\{x_j\}_j^n \subset [0,1]^d$ is a sample set, and $\phi_{n,w}$ is a map $\mathbb{C}^n \to \mathbb{C}$.

For an accuracy $\epsilon \in (0,1)$, the information complexity of numerical integration in a function space $F$ in the random setting is defined by

$$N(\epsilon, F, d) := \inf \left\{ N \in \mathbb{N} \middle| \exists A_{d,N} : E(F, A_{n,w}) \le \epsilon \right\},$$

where

$$E(F, A_{d,N}) := \sup_{\substack{f \in F \\ \|f\|_F \le 1}} \left(\mathbb{E}_\omega \left|I_d(f) - A_{n,w}(f)\right|^2\right)^{1/2}.$$

We say the random algorithms is polynomial tractability, if $N(\epsilon, F, d)$ is polynomial with respect to $d$ and $1/\epsilon$. For further information on tractability in various settings, see the trilogy [36, 37, 38] by Novak and Woźniakowski.

For two nonnegative functions $h, g$ on a common domain, we shall write $h(\tau) = \Theta(g(\tau))$ if there exist absolute constants $C_1, C_2 > 0$ such that $C_1 g(\tau) < h(\tau) < C_2 g(\tau)$ for every $\tau$ in the domain. Also denote by $\mathbb{I}_U(\cdot)$ and $\#U$ the indicator function and the cardinality of $U$, respectively. In this paper, **all hidden constants are absolute constants.** We say that $w \in [-b, b] \,(\mathrm{mod} N)$, meaning that there exists an integer $a$ such that $w + aN \in [-b, b]$. We say that $w \notin [-b, b] \,(\mathrm{mod} N)$, meaning that there does not exist an integer $a$ such that $w + aN \in [-b, b]$. Throughout of this paper, we denote $\mathbb{Z}_N := \mathbb{Z} \cap [0, N)$. For a positive constant $r$ and positive integers $l$ and $N$, we define $G_{r,l}$ and $G_{r,l,\infty}$ as follows

$$G_{r,l} = \frac{1}{r\sqrt{2\pi}} \exp(-\frac{l^2}{2r^2}),\ G_{r,l,\infty} = \sum_{k \in \mathbb{Z}} \frac{1}{r\sqrt{2\pi}} \exp(-\frac{(l+kN)^2}{2r^2}).$$

Next, we need some technical lemmas. We define the median of a set of complex numbers $\{a_j\}_{j=1}^k$ as $median\{a_j\}_{j=1}^k = median\{\Re a_j\}_{j=1}^k + i \times median\{\Im a_j\}_{j=1}^k$, where $median\{\Re a_j\}_{j=1}^k$ and $median\{\Im a_j\}_{j=1}^k$ represent the median of the real part and the imaginary part of $\{a_j\}_{j=1}^k$, respectively. According to Proposition 1 in [32], we have the following corollary.

**Lemma 2.1.** *Let $\mathcal{W}$ be a given function space with semi-norm $\|\cdot\|_{\mathcal{W}}$. If $R_m$ is a random algorithm such that*

$$\sup_{\|f\|_{\mathcal{W}}\le 1} \mathbb{P}\left\{|R_m(f) - \mathrm{INT}(f)| > \epsilon\right\} \le \alpha,$$

*where $0 < \epsilon, \alpha < 1/2$, then for an odd positive integer $k$, it holds*

$$\sup_{\|f\|_{\mathcal{W}}\le 1} \mathbb{P}\left\{\left|median\{R_{m,j}(f)\}_{j=1}^k - \mathrm{INT}(f)\right| > 2\epsilon\right\} \le 2^k \alpha^{k/2},$$

*where $\{R_{m,j}(f)\}_{j=1}^k$ is a set of $k$ independent realizations of $R_m(f)$.*

Based on Claim 7.2 in [23] (see the full version: arXiv:1201.2501v2), we have the following lemma. For the self-containedness of the paper, we provide the proof of the following lemma.

**Lemma 2.2.** *For any $w \in \mathbb{Z}_N$, it holds*

$$\sum_{l\in\mathbb{Z}_N} G_{r,l,\infty} \exp(2\pi i wl/N) = \sum_{k\in\mathbb{Z}} \exp(-2(\pi r(k + w/N))^2), \ \ w \in \mathbb{Z}_N. \tag{6}$$

*Proof.* The proof is carried out entirely in accordance with the proof of Claim 7.2 presented in [23] (see the full version: arXiv:1201.2501v2). We define the Fourier transform over $\mathbb{R}$:

$$\widehat{f}_{\mathbb{R}}(w) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi i w \cdot x) dx.$$

This definition can be extended to the tempered distribution. Let $g_r(x) = \frac{1}{r\sqrt{2\pi}} \exp(-\frac{x^2}{2r^2})$ and $\Delta_1(t)$ denote the *Dirac comb* of period 1: $\Delta_1(t)$ is a Dirac delta function when $t$ is an integer and zero elsewhere. It is well-known that $\widehat{g_r}_{\mathbb{R}}(w) = \exp(-2(\pi r w)^2)$ and $\widehat{\Delta_1}_{\mathbb{R}} = \Delta_1$. Thus,

$$\begin{aligned}
\sum_{l\in\mathbb{Z}_N} G_{r,l,\infty} \exp(2\pi i wl/N) &= \sum_{l\in\mathbb{Z}_N}\sum_{k\in\mathbb{Z}} \frac{1}{r\sqrt{2\pi}} \exp(-\frac{(l+kN)^2}{2r^2}) \exp(2\pi i wl/N) \\
&= \sum_{l\in\mathbb{Z}_N}\sum_{k\in\mathbb{Z}} \frac{1}{r\sqrt{2\pi}} \exp(-\frac{(l+kN)^2}{2r^2}) \exp(2\pi i w(l+kN)/N) \\
&= \sum_{t\in\mathbb{Z}} \frac{1}{r\sqrt{2\pi}} \exp(-\frac{t^2}{2r^2}) \exp(2\pi i wt/N) \\
&= \int_{-\infty}^{\infty} \frac{1}{r\sqrt{2\pi}} \Delta_1(t) \exp(-\frac{t^2}{2r^2}) \exp(2\pi i tw/N) \\
&= \widehat{g_r \cdot \Delta_1}_{\mathbb{R}}(w/N) = (\widehat{g_r}_{\mathbb{R}} * \widehat{\Delta_1}_{\mathbb{R}})(w/N) = (\widehat{g_r}_{\mathbb{R}} * \Delta_1)(w/N) \\
&= \sum_{k\in\mathbb{Z}} \exp(-2(\pi r(k + w/N))^2),
\end{aligned}$$

where the symbol $*$ denotes the convolution operation. $\square$

**Lemma 2.3.** *Let $0 < \epsilon < 1/10$, given a positive integer $B$ greater than 1, we take $r = B\sqrt{\log(1/\epsilon)}$ and $L = \lceil r\sqrt{2\log(1/\epsilon)}\rceil$. By selecting a positive integer $N$ that satisfies $N > 3L$, we have*

$$\sum_{|l|\le L} G_{r,l}\exp(2\pi iwl/N) \le \epsilon \quad \textit{for any} \quad w \notin [-\frac{N}{B}, \frac{N}{B}]\,(\mathrm{mod} N), \tag{7}$$

*and*

$$\left|1 - \sum_{|l|\le L} G_{r,l}\right| \le 10\epsilon. \tag{8}$$

*Proof.* Given that $N > 3L$ and $L = r\sqrt{2\log(1/\epsilon)}$, we have

$$\begin{aligned}
|G_{r,l} - G_{r,l,\infty}| &\le 2\sum_{k=1}^{\infty} \frac{1}{r\sqrt{2\pi}} \exp(-\frac{(2L+(k-1)N)^2}{2r^2}) \\
&\le \frac{2}{r\sqrt{2\pi}} \exp(-\frac{4L^2}{2r^2}) \sum_{k=1}^{\infty} \exp(-\frac{4L(k-1)N}{2r^2}) \\
&\le \frac{\epsilon}{r\sqrt{2\pi}} \exp(-\frac{L^2}{2r^2}) \quad \text{for} \quad l \in [-L, L],
\end{aligned}$$

and utilizing Lemma 2.6 in [4], we obtain

$$\begin{aligned}
\sum_{l\in\mathbb{Z}_N\setminus([0,L]\cup[N-L,N-1])} G_{r,l,\infty} &\le 2\sum_{L<l\le\frac{N}{2}} G_{r,l,\infty} \le 4\sum_{L<l\le\frac{N}{2}} \sum_{k=0}^{+\infty} \frac{1}{r\sqrt{2\pi}} \exp(-\frac{(l+kN)^2}{2r^2}) \\
&\le 4\sum_{L<l\le\frac{N}{2}} \sum_{k=0}^{+\infty} \frac{1}{r\sqrt{2\pi}} \exp(-\frac{l^2+2lkN}{2r^2}) \\
&\le 8\sum_{L<l\le\frac{N}{2}} \frac{1}{r\sqrt{2\pi}} \exp(-\frac{l^2}{2r^2}) \\
&\le 8\epsilon.
\end{aligned}$$

Denote the set $\mathbb{Z}_N\setminus([0,L]\cup[N-L,N-1])$ by $M$, then it follows that,

$$\begin{aligned}
&\left|\sum_{|l|\le L} G_{r,l}\exp(2\pi iwl/N) - \sum_{l\in\mathbb{Z}_N} G_{r,l,\infty}\exp(2\pi iwl/N)\right| \\
=&\left|\sum_{|l|\le L} G_{r,l}\exp(2\pi iwl/N) - \sum_{|l|\le L} G_{r,l,\infty}\exp(2\pi iwl/N) - \sum_{l\in M} G_{r,l,\infty}\exp(2\pi iwl/N)\right| \quad (9) \\
\le&\frac{\epsilon(2L+1)}{r\sqrt{2\pi}} \exp(-\frac{L^2}{2r^2}) + 8\epsilon \le 9\epsilon.
\end{aligned}$$

Using Lemma 2.2 and taking $w = 0$ in the Eq. (6), we have

$$\left|\sum_{l\in\mathbb{Z}_N} G_{r,l,\infty} - 1\right| = \left|\sum_{k\in\mathbb{Z}} \exp(-2(\pi r k)^2) - 1\right| = \left|2\sum_{k=1}^{\infty} \exp(-2(\pi r k)^2)\right| \leq \epsilon. \tag{10}$$

The combination of Eqs. (9) and (10), and the application of triangle inequality yields that

$$\left|1 - \sum_{|l|\leq L} G_{r,l}\right| = \left|1 - \sum_{l\in\mathbb{Z}_N} G_{r,l,\infty} + \sum_{l\in\mathbb{Z}_N} G_{r,l,\infty} - \sum_{|l|\leq L} G_{r,l}\right| \leq 10\epsilon.$$

On the other hand, using the identity (6) and considering the the condition $r = B\sqrt{\log(1/\epsilon)}$, we derive that for $w \notin \left[-\frac{N}{B}, \frac{N}{B}\right] (\mathrm{mod} N)$,

$$\left|\sum_{l\in\mathbb{Z}_N} G_{r,l,\infty} \exp(2\pi i w l/N)\right| = \left|\sum_{k\in\mathbb{Z}} \exp(-2(\pi r(k+w/N))^2)\right| \leq 2\sum_{k=0}^{\infty} \exp(-2(\pi r(k+1/B))^2) \leq \epsilon. \tag{11}$$

This completes the proof. $\square$

The lemma presented below closely resembles Lemma 3.2 in [2] and Lemma 4 in [31].

We shall write $\xi \sim \mathrm{unif}\,\Omega$ to mean that $\xi$ is a random variable having the uniform distribution on the set $\Omega$, and write $\xi_1, \xi_2, \cdots, \xi_m \overset{iid}{\sim} \mathrm{unif}\ \Omega$ to mean that $\xi_1, \xi_2, \cdots, \xi_m$ are drawn i.i.d. from the uniform distribution on $\Omega$.

**Lemma 2.4.** *Let $N$ be a prime number and $B > 1$ be a positive integer. Then, for any $w \in \mathbb{Z}^d$ with $w \neq 0\,(\mathrm{mod} N)$, it holds*

$$H_N \cdot w \in [-N/B, N/B]\,(\mathrm{mod} N),$$

*with probability at most $4/B$ over the randomness of $H_N$, where $H_N := (h_1, h_2, \ldots, h_d)$ is drawn from the uniform distribution over $\mathbb{Z}^d \cap [1, N)^d$.*

*Proof.* Since $w \neq 0\,(\mathrm{mod} N)$, there must be a component whose absolute value is not less than 1. Without loss of generality, let us assume that the $d$-th component $w_d$ satisfies $w_d \neq 0\,(\mathrm{mod} N)$. Then, fixing arbitrary values $h_1, \ldots, h_{d-1}$, with probability at most $4/B$ (no greater than the ratio of the length of the interval $[-N/B, N/B]$ to $2N$) over the randomness of $h_d$, we have

$$\sum_{i=1}^{d} h_i w_i \in [-N/B, N/B]\,(\mathrm{mod} N),$$

thereby completing the proof. $\square$

For any function $f : [0,1]^d \to \mathbb{C}$, we introduce a new function $f_{N,\eta}$ defined as

$$f_{N,\eta}(z/N) := f(z/N + \eta_z), \tag{12}$$

where $z \in \mathbb{Z}_N^d$, and $\eta_z \sim \mathrm{unif}\,[0, 1/N)^d$.

Given a function $f$ with $\|f\|_{L^2([0,1]^d)} \le 1$, we use $a_\xi$ to denote the Fourier coefficient of $f$. Then $f$ can be expressed by

$$f(x) = \sum_{\xi} a_\xi \exp(2\pi i \xi \cdot x).$$

For a fixed integer $\mathcal{M}$, we let $U = \mathbb{Z}^d \cap (-\mathcal{M}, \mathcal{M})^d \backslash \{0\}$, and define $g(x)$ and $R(x)$ as follows:

$$g(x) = \sum_{\xi \in U} a_\xi \exp(2\pi i \xi \cdot x), \;\; R(x) = \sum_{\xi \notin (U \cup \{0\})} a_\xi \exp(2\pi i \xi \cdot x).$$

Then the function $f$ can be rewritten

$$f(x) = a_0 + g(x) + R(x), \quad x \in [0,1]^d. \tag{13}$$

By the definition of $f_{N,\eta}$ in Eq.(12), we obtain that

$$f_{N,\eta}(z/N) = a_0 + g(z/N + \eta) + R(z/N + \eta).$$

For convenience, we let $a_0^* = \widetilde{f}_{N,\eta}^{(N)}(0) = a_0 + \widetilde{g}_{N,\eta}^{(N)}(0) + \widetilde{R}_{N,\eta}^{(N)}(0)$, and

$$R_2(z/N) := R_{N,\eta}(z/N) + g_{N,\eta}(z/N) - g(z/N) - \widetilde{g}_{N,\eta}^{(N)}(0) - \widetilde{R}_{N,\eta}^{(N)}(0). \tag{14}$$

Accordingly, $f_{N,\eta}$ can be rewritten as

$$f_{N,\eta}(z/N) = a_0^* + g(z/N) + R_2(z/N). \tag{15}$$

**Lemma 2.5.** *Let $f$ and $f_{N,\eta}$ be functions defined in* (13) *and* (15)*. Let $K = \#U$. By choosing $N = \Theta(\sqrt{d}K\mathcal{M}/\epsilon)$, we have over the randomness of $\eta$ that it holds*

$$\mathbb{P}\left\{|a_0^* - a_0| \le \epsilon\right\} > 1 - \frac{1}{\epsilon^2 N^d}, \tag{16}$$

*moreover,* $\mathrm{INT}(R_2) = 0$*, and*

$$\mathbb{P}\left\{\frac{1}{N^d}\sum_{z\in\mathbb{Z}_N^d} |R_2(z/N)|^2 \le \frac{1}{\sigma_2}\|R\|_{L^2([0,1]^d)}^2 + \Theta(\epsilon^2)\right\} > 1 - \sigma_2 - \frac{1}{\epsilon^2 N^d}. \tag{17}$$

*Proof.* Since $\mathbb{E}_\eta \widetilde{f}_{N,\eta}^{(N)}(0) = a_0$, we have

$$\mathbb{E}_\eta|\widetilde{f}_{N,\eta}^{(N)}(0) - a_0|^2 = \mathbb{E}_\eta\left|\frac{1}{N^d}\sum_{t\in\{\frac{z}{N}|z\in\mathbb{Z}_N^d\}} f_{N,\eta}(t) - \frac{1}{N^d}\sum_{t\in\{\frac{z}{N}|z\in\mathbb{Z}_N^d\}} \mathbb{E}_\eta f_{N,\eta}(t)\right|^2 \le \frac{1}{N^d}.$$

Then the application of Chebyshev's inequality yields that

$$\mathbb{P}\left\{|\widetilde{g}_{N,\eta}^{(N)}(0) + \widetilde{R}_{N,\eta}^{(N)}(0)| \le \epsilon\right\} = \mathbb{P}\left\{|\widetilde{f}_{N,\eta}^{(N)}(0) - a_0| \le \epsilon\right\} > 1 - \frac{1}{\epsilon^2 N^d}. \tag{18}$$

By an direct calculation, we obtain

$$\mathbb{E}_\eta \frac{1}{N^d} \sum_{z\in\mathbb{Z}_N^d} |R_{N,\eta}(z/N)|^2 = \|R\|^2_{L^2([0,1]^d)}.$$

Then for a positive constant $\sigma_2$, the application of Markov's inequality yields that

$$\mathbb{P}\left\{ \frac{1}{N^d} \sum_{z\in\mathbb{Z}_N^d} |R_{N,\eta}(z/N)|^2 \leq \frac{1}{\sigma_2}\|R\|^2_{L^2([0,1]^d)} \right\} > 1-\sigma_2. \tag{19}$$

Finally, observing that

$$\begin{aligned} |g_{N,\eta}(z/N) - g(z/N)| &= \left| \sum_{\xi\in U} (a_\xi \exp(2\pi i \xi \cdot \eta_z) - a_\xi) \exp(2\pi i \xi \cdot z/N) \right| \\ \leq \Theta(K \sup_{\xi\in U} \|\xi\|\|\eta_z\|) &\leq \Theta(K\sqrt{d}\mathcal{M}/N) \leq \Theta(\epsilon). \end{aligned} \tag{20}$$

The inequality (17) can be verified by eqs. (14) (18), (19) and (20). It ends the proof. $\square$

To establish the theoretical analysis framework for our algorithm, we meticulously designed a function space that is comprised of the function $f$ specifically defined in (13):

$$\begin{aligned} F_{K,\mathcal{M},\lambda_1,\lambda_2} := \Bigg\{ & f(x) = a_0 + g(x) + R(x), x \in [0,1]^d \\ \Bigg| & g(x) = \sum_{\xi\in v_\mathcal{M}} a_\xi \exp(2\pi i \xi \cdot x), v_\mathcal{M} := \mathbb{Z}^d \cap (-\mathcal{M},\mathcal{M})^d \backslash \{0\}, \sum_{j\geq 1} |a_j| \leq \lambda_1, \\ & \# v_\mathcal{M} = K, R(x) = \sum_{w\in\mathbb{Z}^d\backslash(v_\mathcal{M}\cup\{0\})} b_j \exp(2\pi i w \cdot x), \|R\|^2_{L^2([0,1]^d)} \leq \lambda_2 \Bigg\}. \end{aligned} \tag{21}$$

It is trivial to prove that $\mathrm{INT}(f) = a_0$ for any $f \in F_{K,\mathcal{M},\lambda_1,\lambda_2}$.

**Theorem 2.6.** *Let $f \in F_{K,\mathcal{M},\lambda_1,\lambda_2}$ with $\|f\|_{L^2([0,1]^d)} \leq 1$ and $\lambda_1, \lambda_2 \geq 1$. Given $0 < \sigma, \sigma_2 < 1/10$, $n \geq K+1$, $0 < \epsilon < \sigma/n$, we take $B = \lceil 4n/\sigma \rceil$, $r = B\sqrt{\log(1/\epsilon)}$, $L = \lceil r\sqrt{2\log(1/\epsilon)} \rceil$, and let $N = \max\{\Theta(1/(\sqrt{\sigma}\epsilon)^{2/d}), \Theta(\sqrt{d}K\mathcal{M}/\epsilon), 3L\}$ be a prime number. Then it holds that over the randomness of $H_N$, $z$ and $\eta$*

$$\mathbb{P}\left\{ \left| I(f_{N,\eta})_{H_N,L,r,z} - \mathrm{INT}(f) \right| \leq \Theta(\lambda_1\epsilon + \sqrt{\frac{\lambda_2}{n\sigma_2}}) \right\} > 1 - \sigma - \sigma_2. \tag{22}$$

*Taking the median of outputs $\{I(f_{N,\eta})_{H_{N_j},L,r,z_j}\}_{j=1}^t$, which are obtained by independently randomizing $H_N$, $z$ and $\eta$ for $t$ times, here $t$ is an odd number greater than 7, then we have*

$$\mathbb{P}\left\{ \left| median\{I(f_{N,\eta_j})_{H_{N_j},L,r,z_j}\}_{j=1}^t - \mathrm{INT}(f) \right| \leq \Theta(\lambda_1\epsilon + \sqrt{\frac{\lambda_2}{n\sigma_2}}) \right\} > 1 - (2\sqrt{\sigma+\sigma_2})^t. \tag{23}$$

*Furthermore, the RMSE of the presented integration method is bounded by*

$$\begin{aligned}&\bigg(\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t}\big|median\{I(f_{N,\eta_j})_{H_{N_j},L,r,z_j}\}_{j=1}^t-\mathrm{INT}(f)\big|^2\bigg)^{1/2}\\ &\leq\Theta\bigg(\lambda_1+\sqrt{\lambda_2/(n\sigma_2)}+(2\sqrt{\sigma+\sigma_2})^t+(2/\sqrt{100})^t\bigg).\end{aligned}\tag{24}$$

*Proof.* For any function $f\in F_{K,\mathcal{M},\lambda_1,\lambda_2}$, $f$ can be can be expressed by

$$f(x)=a_0+g(x)+R(x),\tag{25}$$

with the conditions

$$\sum_{j\geq 1}|a_j|\leq\lambda_1,\mathrm{INT}(R)=0,\ \text{and}\ \|R\|^2_{L^2([0,1]^d)}\leq\lambda_2.$$

As presented in (15), we have

$$f_{N,\eta}(z/N)=a_0^*+g(z/N)+R_2(z/N),\quad z\in\mathbb{Z}_N^d.\tag{26}$$

By choosing a prime number $N=\Theta(\max\{1/(\sqrt{\sigma}\epsilon)^{2/d}),\sqrt{d}K\mathcal{M}/\epsilon)$ and using Lemma 2.5, the following inequality holds with probability at least $1-\sigma$ over the randomness of $\eta$,

$$|a_0^*-a_0|\leq\Theta(\epsilon),\tag{27}$$

and the following inequality holds with probability at least $1-\sigma-\sigma_2$,

$$\frac{1}{N^d}\sum_{z\in\mathbb{Z}_N^d}|R_2(z/N)|^2=\sum_{w\in\mathbb{Z}_N^d}|\widetilde{R_2}^{(N)}(w)|^2\leq\frac{\|R\|^2_{L^2([0,1]^d)}}{\sigma_2}+\Theta(\epsilon^2)\leq\frac{\lambda_2}{\sigma_2}+\Theta(\epsilon^2).\tag{28}$$

For the case when the frequency $\xi$ or $w$ equals 0, by using Eq.(26) and Lemma 2.5, it holds that

$$\widetilde{f}_{N,\eta}^{(N)}(0)=a_0^*,\ \widetilde{g}^{(N)}(0)=0,\ \text{and}\ \widetilde{R}_2^{(N)}(0)=0.$$

From Lemma 2.3 and Eq.(27), with probability at least $1-\sigma$ , we deduce

$$|\sum_{|l|\leq L}a_0^*G_{r,l}-a_0|\leq\Theta(\epsilon).\tag{29}$$

We will analyze the last two terms on the right side of Eq. (26) separately. Let $n\geq K+1$, $B=\lceil 4n/\sigma\rceil$, $r=B\sqrt{\log(1/\epsilon)}$, and $L=r\sqrt{2\log(1/\epsilon)}$. By Lemma 2.4, it follows

$$\mathbb{P}\bigg\{H_N\cdot w^{(j)}\in\bigg[-\frac{N}{B},\frac{N}{B}\bigg](\mathrm{mod}N)\bigg\}\leq\sigma/n,\ 1\leq j\leq K,$$

which leads to

$$\mathbb{P}\bigg\{H_N\cdot w^{(j)}\notin\bigg[-\frac{N}{B},\frac{N}{B}\bigg](\mathrm{mod}N)\text{ holds for every }j=1,2,\ldots,K\bigg\}\geq 1-\sigma.$$

Furthermore, by Lemma 2.3, it holds

$$\mathbb{P}\bigg\{ \sum_{|l|\le L} g\bigg(\bigg\{\frac{z-lH_N}{N}\bigg\}\bigg) G_{r,l} \le \lambda_1 \epsilon \text{ holds for every } z\in\mathbb{Z}_N^d \bigg\} \ge 1-\sigma. \tag{30}$$

Next, we analyze $R_2$. For each $\xi \in \mathbb{Z}_N^d \backslash \{0\}$, we let $\widetilde{R_2}^{(N)}(\xi) = c_\xi$ and denote

$$t_\xi := \frac{1}{N^d} \sum_{z\in[0,N)^d} \bigg| \sum_{|l|\le L} c_\xi \exp\bigg(2\pi i \xi \cdot \bigg\{\frac{z-lH_N}{N}\bigg\}\bigg) G_{r,l}\bigg|^2,$$

and

$$U_{H_N,B} := \bigg\{ \xi | H_N\cdot \xi \in \big[-\frac{N}{B}, \frac{N}{B}\big] (\text{mod} N) \bigg\}.$$

The application of Lemma 2.3 leads to $\sum G_{r,l} \le 1+2\epsilon \le 2$, therefore, we obtain that

$$t_\xi \le \frac{1}{N^d} \sum_{z\in[0,N)^d} \bigg| \sum_{|l|\le L} |c_\xi| G_{r,l} \bigg|^2 \le \Theta(|c_\xi|^2). \tag{31}$$

In addition, by using Lemma 2.4, we derive

$$\mathbb{P}\left\{\xi \in U_{H_N,B}\right\} \le \Theta(1/B). \tag{32}$$

The combination of Eqs. (31) and (32) results in

$$\mathbb{E}_{H_N}\big(t_\xi | \xi \in U_{H_N,B}\big) \mathbb{P}\left\{\xi\in U_{H_N,B}\right\} \le \Theta\bigg(\frac{|c_\xi|^2}{B}\bigg).$$

On the ther hand, by using Eq.(7), we have $\mathbb{E}_{H_N}\big(t_\xi|\xi\notin U_{H_N,B}\big) \le \Theta(|c_\epsilon|^2\epsilon^2)$. Thus,

$$\mathbb{E}_{H_N}\big(t_\xi | \xi \notin U_{H_N,B}\big) \mathbb{P}\left\{\xi\notin U_{H_N,B}\right\} \le \Theta\bigg(\epsilon^2 |c_\xi|^2\bigg).$$

Therefore, it follows that for $H_N \sim \text{unif}\, \mathbb{Z}^d \cap [1,N)^d$.

$$\begin{aligned} \mathbb{E}_{H_N} t_\xi &= \mathbb{E}_{H_N}\big(t_\xi | \xi \in U_{H_N,B}\big) \mathbb{P}\left\{\xi\in U_{H_N,B}\right\} + \mathbb{E}_{H_N}\big(t_\xi | \xi \notin U_{H_N,B}\big) \mathbb{P}\left\{\xi\notin U_{H_N,B}\right\} \\ &\le \Theta\bigg(\frac{|c_\xi|^2}{B} + \epsilon^2|c_\xi|^2\bigg). \end{aligned} \tag{33}$$

Upon observing that for every distinct $\xi, \eta \in \mathbb{Z}_N^d$,

$$\sum_{z\in\mathbb{Z}_N^d} \bigg( \sum_{|l|\le L} \exp\bigg(2\pi i \xi \cdot \bigg\{\frac{z-lH_N}{N}\bigg\}\bigg) G_{r,l} \bigg) \bigg( \sum_{|l|\le L} \exp\bigg(2\pi i \eta \cdot \bigg\{\frac{z-lH_N}{N}\bigg\}\bigg) G_{r,l} \bigg) = 0. \tag{34}$$

and $\widetilde{R_2}^{(N)}(\xi) = c_\xi$, by combining Eqs. (28, 33), we immediately arrive at the conclusion that with probability at least $1-\sigma-\sigma_2$ over the randomness of $\eta$,

$$\mathbb{E}_{H_N}\mathbb{E}_z\left|\sum_{|l|\le L} R_2\left(\left\{\frac{z-lH_N}{N}\right\}\right)G_{r,l}\right|^2 = \mathbb{E}_{H_N}\sum_{\xi\in\mathbb{Z}^d\cap[0,N]^d\setminus\{0\}} t_\xi \le \Theta\left(\frac{\lambda_2}{B\sigma_2}+(1+\lambda_2/\sigma_2)\epsilon^2\right), \tag{35}$$

where $z\sim$ unif $\mathbb{Z}_N^d$.

Recalling the conditions $\epsilon < \sigma/n$ and $B=\lceil 4n/\sigma\rceil$, we obtain the following inequality by applying Markov's inequality,

$$\mathbb{P}\left\{\left|\sum_{|l|\le L} R_2\left(\left\{\frac{z-lH_N}{N}\right\}\right)G_{r,l}\right|^2 \le \Theta\left(\frac{\lambda_2}{n\sigma_2}\right)\right\} \ge (1-\sigma)(1-\sigma-\sigma_2)\ge 1-2\sigma-\sigma_2, \tag{36}$$

where the probability determined by the random variables $z$, $H_N$ and $\eta$. Summarizing Eqs. (26, 29, 30, 36), and applying Markov's Inequality, we obtain

$$\mathbb{P}\left\{\left|\sum_{|l|\le L} f\left(\left\{\frac{z-lH_N}{N}\right\}\right)G_{r,l}-a_0\right| \le \Theta(\lambda_1\epsilon+\sqrt{\lambda_2/(n\sigma_2)})\right\}\ge 1-4\sigma-\sigma_2.$$

Replacing $\sigma$ by $\sigma/4$, we complete the proof of Eq.(22).

By randomly selecting $H_N$, $z$ and $\eta$ to compute $I(f_{N,\eta})_{H_N,L,r,z}$ $t$ times, we obtain $t$ different results. Extracting the median of the obtained results, the estimation (23) can be naturally derived from Lemma 2.1. Denote the median by $Y$. Then

$$\begin{aligned}
&\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t}\left|median\{I(f_{N,\eta_j})_{H_{N_j},L,r,z_j}\}_{j=1}^t-\mathrm{INT}(f)\right|^2\\
\le&\sum_{k\ge 0}\mathbb{P}\{200k\le|Y|\le 200(k+1)\}\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t}\left(\left|Y-\mathrm{INT}(f)\right|^2\middle|200k\le|Y|\le 200(k+1)\right)\\
=:&\sum_{k\ge 0}E_k.
\end{aligned} \tag{37}$$

Combining $|\mathrm{INT}(f)|\le 1$ and Eq. (23), we obtain that

$$E_0\le\left(\Theta(\lambda_1\epsilon+\sqrt{\lambda_2/(n\sigma_2)})(1-(2\sqrt{\sigma+\sigma_2})^t)+(200+1)(2\sqrt{\sigma+\sigma_2})^t\right)^2.$$

Notice

$$\begin{aligned}
\mathbb{E}_\eta\mathbb{E}_z|I(f_{N,\eta})_{H_N,L,r,z}| &\le \sum_{|l|\le L} G_{r,l}\mathbb{E}_\eta\mathbb{E}_z|f_{N,\eta}(\{\frac{z-lH_N}{N}\})|\\
&=\left(\sum_{|l|\le L} G_{r,l}\right)\mathbb{E}_\eta\mathbb{E}_z|f_{N,\eta}(z/N)| = \left(\sum_{|l|\le L} G_{r,l}\right)\|f\|_{L^1([0,1]^d)}\le 2\|f\|_{L^2([0,1]^d)}\le 2,
\end{aligned}$$

Thus, for $k\ge 1$, using Markov's Inequality, we have

$$\mathbb{P}\{|I(f_{N,\eta})_{H_N,L,r,z}|\ge 200k\}\le\frac{1}{100k},$$

it follows that

$$\mathbb{P}\{|Y| \geq 200k\} \leq \binom{t}{\frac{t+1}{2}}\left(\frac{1}{100k}\right)^{\frac{t+1}{2}} \leq (2/\sqrt{100})^t/k^{\frac{t+1}{2}}.$$

Therefore,

$$E_k \leq (200(k+2))^2(2/\sqrt{100})^t/k^{\frac{t+1}{2}}, \quad \text{for} \quad k \geq 1.$$

Given that $t \geq 7$, there exists an absolute constant $C$ such that

$$\sum_{k\geq 1}(k+2)^2/k^{\frac{t+1}{2}} \leq C.$$

Consequently,

$$\sum_{k\geq 1} E_k \leq \Theta((2/\sqrt{100})^t),$$

and

$$\begin{aligned}&\left(\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t}\left|median\{I(f_{N,\eta})_{H_{N_j},L,r,z_j}\}_{j=1}^t - \mathrm{INT}(f)\right|^2\right)^{1/2}\\ &\leq (\sum_{k\geq 0} E_k)^{1/2} \leq \Theta(\lambda_1\epsilon + \sqrt{\lambda_2/(n\sigma_2)} + (2\sqrt{\sigma+\sigma_2})^t + (2/\sqrt{100})^t).\end{aligned} \tag{38}$$

This completes the proof. $\square$

We are now in position to formulate an accurate estimation for the unit ball of periodic isotropic Sobolev space

$$H^s(\mathcal{T}^d) := \left\{ f \in L^2(\mathcal{T}^d) \middle| \|f\|^2_{H^s(\mathcal{T}^d))} := \sum_{w=(w_1,\ldots,w_d)\in\mathbb{Z}^d} (1 + (\sum_{j=1}^d |2\pi w_j|)^s)^2 |\hat{f}(w)|^2 \leq 1 \right\},$$

where $\mathcal{T} = \mathbb{R}/\mathbb{Z} = [0,1)$.

**Theorem 2.7.** *Let $f \in H^s(\mathcal{T}^d)$ with $s \geq 0$. Given $0 < \sigma < 1/10$. let $\epsilon < \sigma/n$, $N = \max\{\Theta(1/(\sqrt{\sigma}\epsilon)^{2/d}), \Theta(\sqrt{d}B^{1+1/d}/\epsilon)\}$ be a prime number, $n > 0$, $B = \lceil 4n/\sigma \rceil$, $r = B\sqrt{\log(1/\epsilon)}$, $L = \lceil r\sqrt{2\log(1/\epsilon)}\rceil$, and assume $L < N/3$. It holds with probability at least $1 - \sigma - 1/10$ over the randomness of $H_N$, $z$ and $\eta$ that*

$$\left| I(f_{N,\eta})_{H_N,L,r,z} - \mathrm{INT}(f)\right| \leq \Theta\left(\frac{1}{n^{s/d+1/2}} + n\epsilon\right). \tag{39}$$

*By obtaining the $median\{I(f_{N,\eta_j})_{H_{N_j},L,r,z_j}\}_{j=1}^t$ as in Theorem 2.6, we have the following holds with probability at least $1 - (2\sqrt{\sigma + 1/10})^t$,*

$$\left|median\{I(f_{N,\eta_j})_{H_{N_j},L,r,z_j}\}_{j=1}^t - \mathrm{INT}(f)\right| \leq \Theta\left(\frac{1}{n^{s/d+1/2}} + n\epsilon\right). \tag{40}$$

*Furthermore, the RMSE is bounded by*

$$\begin{aligned}&\left(\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t}\left|median\{I(f_{N,\eta})_{H_{N_j},L,r,z_j}\}_{j=1}^t-\mathrm{INT}(f)\right|^2\right)^{1/2}\\&\qquad\leq\Theta\left(\frac{1}{n^{s/d+1/2}}+n\epsilon+(2\sqrt{\sigma+1/10})^t\right).\end{aligned}\tag{41}$$

**Remark 2.8.** *Form Eq.(5) and Eq. (41), we know that the sample size is $M := tL$. Choosing $\sigma = 1/10$, $t = \log_{\sqrt{5}/2}(n^{s/d+1/2})$ and $\epsilon = 1/n^{s/d+3/2}$ in Eq. (41), we know that the upper bound of the RMSE error is*

$$\Theta((3/2+s/d)^{s/d+3/2}(\log M)(\log M/M)^{s/d+1/2}).\tag{42}$$

*Let Eq. (42) be equal to $\epsilon$, and we know that the sample size $M$ is polynomial with respect to $d$ and $1/\epsilon$. Therefore, the random algorithm is polynomial tractability. The upper bound (42) (up to an absolute constant factor) is also holds for the integration in isotropic Sobolev space with compact support, see Corollary 2.9 and Remark 2.10. From [30, 43], we know that the upper bounds of their integration algorithm for the isotropic Sobolev space with compact support not less than $2^{\Theta(d)}M^{-s/d-1/2}$, thus our upper bound is superior to that presented in [30, 43] when $s \leq \Theta(d)$ and $M \leq 2^{2^{\Theta(d)}}$ (namely, $\log^{\Theta(1)} M \leq 2^{\Theta(d)}$). Moreover, if we know that the upper bound of the order $s$, then the algorithm is universal for $s \leq S$, where $S$ denotes the upper bound of $s$. That is, it suffices to set the number of repetitions $t = \log_{\sqrt{5}/2}(n^{S/d+1/2})$ and the minimum precision parameter (see Theorem 2.7) $\epsilon = 1/n^{S/d+3/2}$ (the corresponding sample cost is $\Theta(\log(1/\epsilon))$) without the need to ascertain the specific value of $s$.*

*Proof.* Let $n$ be a positive number, and define $\beta_{n,d} = \frac{n^{1/d}}{2\pi}$. For any $f \in H^s(\mathcal{T}^d)$, we decompose $f$ into the following form:

$$f(x) = a_0 + g(x) + R(x),$$

where

$$g(x) = \sum_{w\in\mathbb{Z}^d\cap(-\beta_{n,d},\beta_{n,d})^d\backslash\{0\}} a_w \exp(2\pi i w\cdot x),$$

and

$$R(x) = \sum_{w\notin\mathbb{Z}^d\cap(-\beta_{n,d},\beta_{n,d})^d} a_w \exp(2\pi i w\cdot x).$$

Since $f \in H^s(\mathcal{T}^d)$, we have $\|R\|_{L^2([0,1]^d)} \leq \Theta(n^{-\frac{s}{d}}\|f\|_{H^s}) = \Theta(n^{-\frac{s}{d}})$ (since $\|f\|_{H^s} \leq 1$, see definition of $H^s(\mathcal{T}^d)$) and

$$\#\{w|w\in\mathbb{Z}^d\cap(-\beta_{n,d},\beta_{n,d})^d\backslash\{0\}\}\leq n.$$

Let $N = \Theta(1/(\sqrt{\sigma}\epsilon)^{2/d}))$ and $B = 4n/\sigma$. Theorem 2.7 follows directly from Theorem 2.6 (choosing $\sigma_2 = 1/10$, and $\mathcal{M} = \beta_{n,d}$ in Theorem 2.6). **Note that:** If $n \leq 5\pi$, $\#\{w|w \in \mathbb{Z}^d \cap (-\beta_{n,d}, \beta_{n,d})^d\backslash\{0\}\} = 0$, this case does not affect the application of Theorem 2.6 to prove Theorem 2.7, since we just note the condition $n \geq K + 1$ and the upper bound $\Theta(\lambda_1\epsilon + \sqrt{\frac{\lambda_2}{n\sigma_2}})$ in Theorem 2.6. $\square$

Given a bounded measurable set $\Omega$ with volume 1, we denote the unit ball of the isotropic Sobolev spaces with compact support over $\Omega$ by $\mathring{H}^s(\Omega)$ (the definition see [43]). Here the assumption $|\Omega| = 1$ aligns with the specifications outlined in [43]. For any $f \in \mathring{H}^s(\Omega)$, we can define

$$F(x) = \sum_{k \in \mathbb{Z}^d} f(k + x).$$

It is noteworthy that

$$\int_\Omega f(x)dx = \int_{\mathcal{T}^d} F(x)dx \quad \text{and} \quad \|F\|_{H^s(\mathcal{T}^d)} \leq 1.$$

Thus the error bounds (39), (40), and (41) stated previously remain valid for $\mathrm{INT}(F) = \int_\Omega f(x)dx$. A pertinent question arises regarding the number of samples required to determine the value of $F(x)$. More precisely, based on Eq. (5), how many samples are necessary to obtain the values of the set $\{F_{N,\eta}(\{\frac{z-lH_N}{N}\})\}_{l=-L}^{L}$? Let us define

$$\mathbb{J}^\Omega(x) := \sum_{k \in \mathbb{Z}^d} \mathbb{I}_\Omega(k + x),$$

where $\mathbb{I}_\Omega$ is the indicator function of $\Omega$. With a certain probability, we only require $\Theta(L)$ samples. In fact,

$$\begin{aligned}
2L + 1 = (2L+1)\text{volume}(\Omega) &= (2L+1)\int_\Omega \mathbb{I}_\Omega(x)dx \\
= (2L+1)\int_{[0,1]^d} \mathbb{J}^\Omega(x)dx &= \sum_{l=-L}^{L} \mathbb{E}_{\eta_l}\mathbb{E}_z \mathbb{J}^\Omega_{N,\eta_l}(\frac{z - lH_N}{N}) \\
&= \mathbb{E}_{\eta_1}\mathbb{E}_{\eta_2}\cdots\mathbb{E}_{\eta_l}\mathbb{E}_z \sum_{l=-L}^{L} \#\{k \big| k \in \mathbb{Z}^d, (z - lH_N)/N + \eta_{l,z} + k \in \Omega\},
\end{aligned}$$

where $z \sim \text{unif}\, \mathbb{Z}^d \cap [0, N)^d$ and $\{\eta_{l,z}\}_{l=-L}^{L} \overset{iid}{\sim} \text{unif}\,[0, 1/N)^d$ (see Eq.(12)). Then by Markov's inequality,

$$\mathbb{P}\left\{\sum_{l=-L}^{L} \#\{k \big| k \in \mathbb{Z}^d, (z - lH_N)/N + \eta_l + k \in \Omega\} \geq 200L + 100\right\} \leq 1/100.$$

In other words, with a probability of at least 99/100, the number of samples required to obtain $\left\{F_{N,\eta}(\{\frac{z-lH_N}{N}\})\right\}_{l=-L}^{L}$ is at most $200L + 100$.

When searching the sets

$$V_l := \{z - lH_N + \eta_l + k \big| k \in \mathbb{Z}^d, z - lH_N + \eta_l + k \in \Omega\}, -L \leq l \leq L,$$

we do not require any samples of the function $f$. Upon discovering that the number of elements in the union $\cup_{-L \leq l \leq L} V_l$ exceeds $200L+100$, we promptly output the result (denoted as $I^*(F_{N,\eta})_{H_N,L,r,z}$) as zero. Otherwise, we set $I^*(F_{N,\eta})_{H_N,L,r,z} = I(F_{N,\eta})_{H_N,L,r,z}$.

Therefore, for the function $F$, the probability of success for Eq. (39) , where $f$ is replaced with $F$, with a required sample size of at most $200L+100$ . This ensures that the sample complexity of our algorithm remains polynomial with respect to $d$. Consequently, we have the following corollary.

**Corollary 2.9.** *Let $f \in \mathring{H}^s(\Omega)$ and define $F(x) = \sum_{k\in\mathbb{Z}^d} f(k+x)$. Assume the same parameter settings as those in Theorem 2.7. Then it holds with probability at least $1-\sigma-1/10-1/100$ over the randomness of $H_N$, $z$ and $\eta$ that*

$$\left| I^*(F_{N,\eta})_{H_N,L,r,z} - \mathrm{INT}(F) \right| \leq \Theta\left( \frac{1}{n^{s/d+1/2}} + n\epsilon \right).$$

*By obtaining the median$\{I^*(F_{N,\eta_j})_{H_{N_j},L,r,z_j}\}_{j=1}^t$ as in Theorem 2.6, we have the following holds with probability at least $1-(2\sqrt{\sigma+1/10+1/100})^t$,*

$$\left| median\{I^*(F_{N,\eta_j})_{H_{N_j},L,r,z_j}\}_{j=1}^t - \mathrm{INT}(F) \right| \leq \Theta\left( \frac{1}{n^{s/d+1/2}} + n\epsilon \right).$$

*Furthermore, the RMSE is bounded by*

$$\begin{aligned} &\left( \mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t} \left| median\{I^*(F_{N,\eta})_{H_{N_j},L,r,z_j}\}_{j=1}^t - \mathrm{INT}(F) \right|^2 \right)^{1/2} \\ &\quad \leq \Theta\left( \frac{1}{n^{s/d+1/2}} + n\epsilon + (2\sqrt{\sigma+1/10+1/100})^t \right). \end{aligned}$$

**Remark 2.10.** *We define the isotropic Sobolev space in accordance with the definition provided in [43], albeit with minor variations from the one presented in [30]. Nevertheless, the conclusion remains valid for the isotropic Sobolev space with compact support defined in [30].*

**Remark 2.11.** *The sample size is $M := (200L+100)t$. The analysis of the sample complexity and polynomial tractability of the algorithm is analogous to Theorem 2.7, as detailed in Remark 2.8 .*

# 3 Integration of Analytic Functions and the Wiener-type Functions

We present the semi-exponential convergence rate estimates of numerical integration for analytic functions in an asymptotic sense. Unlike the previous section, we do not endeavor to demonstrate that the algorithm possesses polynomial tractability simultaneously when it achieves the semi-exponential convergence rate.

**Theorem 3.1.** *Let $f$ be analytic over $[0,1]^d$. Let $0<\sigma<1/10$, $n>2$, $0<\epsilon<\sigma/n$, $B = \lceil \frac{4n}{\sigma} \rceil$, $r = B\sqrt{\log(\frac{1}{\epsilon})}$, $L = \lceil r\sqrt{2\log(\frac{1}{\epsilon})} \rceil$. Also let $N = \max\{\Theta(\frac{1}{(\sqrt{\sigma}\epsilon)^{\frac{2}{d}}}), \Theta(\frac{\sqrt{d}B^{1+\frac{1}{d}}}{\epsilon}), 3L\}$*

*be a prime number. Then it holds with probability at least $1-\sigma-1/(2\ln L)$ over the randomness of $H_N$, $z$ and $\eta$ that*

$$\left|I(f_{N,\eta})_{H_N,L,r,z}-\mathrm{INT}(f)\right|\leq C_{f,0}(\log(L)\exp(-\frac{n^{C_*}}{dC_{f,*}})+n\epsilon), \tag{43}$$

*where $C_{f,0}$ is some constants depend on $f$ and $d$, $C_{f,*}$ are some constants depend on $f$ and $C_*$ is some absolute constant. With $median\{I(f_{N,\eta_j})_{H_{N_j},L,r,z_j}\}_{j=1}^t$ as in Theorem 2.6, it holds with probability at least $1-(2\sqrt{\sigma+1/(2\ln L)})^t$ that*

$$\left|median\{I(f_{N,\eta_j})_{H_{N_j},L,r,z_j}\}_{j=1}^t-\mathrm{INT}(f)\right|\leq C_{f,0}(\log(L)\exp(-\frac{n^{C_*}}{dC_{f,*}})+n\epsilon). \tag{44}$$

*Furthermore,*

$$\begin{aligned}&\left(\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t}\left|median\{I(f_{N,\eta})_{H_{N_j},L,r,z_j}\}_{j=1}^t-\mathrm{INT}(f)\right|^2\right)^{1/2}\\&\qquad\leq C_{f,0}(\log(L)\exp(-\frac{n^{C_*}}{dC_{f,*}})+n\epsilon)+(2\sqrt{\sigma+1/(2\ln L)})^t.\end{aligned} \tag{45}$$

*Proof.* We first explain our proof strategy. For $f$, we use the trick of change of variable (see [43, 34, 42]) to construct a smooth function $Tf$ that vanish on the boundary of $[0,1]^d$. On the other hand, we know from the results in [41] that there exist constants $C_{f,1}>1$ and $C_{f,2}>1$ depend only on $f$, such that the following equation holds.

$$\left|\frac{\partial^{n_1+\cdots+n_d}f}{\partial x_1^{n_1}\ldots\partial x_s^{n_d}}(t)\right|\leq C_{f,1}(C_{f,2})^n n! \quad \text{for} \quad t\in[0,1]^d, \tag{46}$$

where $n=n_1+\cdots+n_d$. Finally, we derive an estimate for the Fourier coefficients of $TF$, leveraging Theorem 2.6, we can establish the proof of Theorem 3.1.

Subsequently, we present the details of the proof. We consider the following function introduced in [34]

$$\psi(y)=\begin{cases}\int_0^y\exp(-1/(\xi(1-\xi)))\,\mathrm{d}\xi/\int_0^1\exp(-1/(\xi(1-\xi)))\,\mathrm{d}\xi : y\in[0,1]\\ 1 : y>1\\ 0 : y<0\end{cases} \tag{47}$$

For $t=(t_1,t_2,\ldots,t_d)\in\mathbb{R}^d$, let $f_\psi(t)=f(\psi(t_1),\psi(t_2),\ldots,\psi(t_d))$, and

$$Tf(t):=\left|\prod_{j=1}^d\psi^{'}(t_j)\right|f_\psi(t)=\prod_{j=1}^d\psi^{'}(t_j)f_\psi(t),$$

then it is obvious that $\int_{[0,1]^d}f(t)dt=\int_{[0,1]^d}Tf(t)dt$.

Next, we estimate$\left|\frac{\partial^k(Tf)}{\partial t_j^k}(t)\right|$. Observe that

$$\sup_{y\in(0,1)}\left|\frac{d^k\psi}{dy^k}(y)\right|=\sup_{y\in(0,1/2]}\left|\frac{d^k\psi}{dy^k}(y)\right|\leq\sup_{y\in(0,1/2]}\frac{c_1 2^k}{y^k}\exp(\frac{-1}{y})=c_1(2k/e)^k, \tag{48}$$

where $c_1 \leq 2/\int_0^1 \exp(-1/(\xi(1-\xi)))$ is an absolute constant. To obtain the last equality in the Eq. (48), it suffices to observe that $k$ is a critical point of the function $x^k/e^x, x \in (1, +\infty)$. From Eq.(46) and Eq. (48), we have

$$\left|\frac{\partial^k f_\psi}{\partial t_j^k}(t)\right| \leq 2^k C_{f,1}(C_{f,2})^k k! c_1 (2k/e)^k \quad \text{for} \quad \text{every} \quad t \in [0,1]^d, \quad k \in \mathbb{N}. \tag{49}$$

From Eq.(48) and Eq. (49), using Leibniz's formula and Stirling's formula, we know that there exist absolute constants $C_1$ and $C_2$ such that

$$\left|\frac{\partial^k (Tf)}{\partial t_j^k}(t)\right| \leq C_{f,1}(C_1 C_{f,2} k)^{C_2(k+d)} \quad \text{for} \quad \text{every} \quad t \in [0,1]^d, \quad k \in \mathbb{N} \quad \text{and} \quad 1 \leq j \leq d. \tag{50}$$

Using Eq.(50), and integral by parts, we have

$$|\widehat{Tf}(w)| \leq C_{f,1}(C_1 C_{f,2} k)^{C_2(k+d)}/\|w\|_\infty^k,$$

where $w = (w_1, w_2, \ldots, w_d) \in \mathbb{N}_0^d \backslash \{0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\|w\|_\infty := \max_{1\leq j\leq d}\{|w_1|, |w_2|, \ldots, |w_d|\}$. Choosing $k = \lfloor \frac{(\|w\|_\infty/e)^{1/(2C_2)}}{C_1 C_{f,2}} \rfloor$ and $k \geq d$, we have

$$|\widehat{Tf}(w)| \leq C_{f,1} e^{-\lfloor \frac{(\|w\|_\infty/e)^{1/(2C_2)}}{C_1 C_{f,2}} \rfloor} \quad \text{for} \quad \frac{(\|w\|_\infty/e)^{1/(2C_2)}}{C_1 C_{f,2}} \geq d+1.$$

Therefore, there exists the constants $C_3$ and $C_{f,3}$ such that

$$|\widehat{Tf}(w)| \leq C_{f,1} e^{-\frac{\|w\|_\infty^{C_3}}{C_{f,3}}} \quad \text{for} \quad \frac{\|w\|_\infty^{C_3}}{C_{f,3}} \geq d+1$$

Let $\beta, \rho > 1, \lambda > 0$, we observe the fact that

$$\sum_{m=1}^\infty m^d \rho^{-(m\beta)^\lambda} = \rho^{-(\beta)^\lambda} \sum_{m=1}^\infty m^d \rho^{-((m\beta)^\lambda - \beta^\lambda)} \leq \rho^{-(\beta)^\lambda} \sum_{m=1}^\infty m^d \rho^{-(m^\lambda - 1)} = C_{\rho,\lambda,d}\rho^{-(\beta)^\lambda},$$

where the constant $C_{\rho,\lambda,d} := \sum_{m=1}^\infty m^d \rho^{-m^\lambda+1}$. Therefore, for $\beta \in \mathbb{N}$, there exists the constant $C_{f,4,d}$ depend on $f$ and $d$, the constant $C_{f,5}$ depend on $f$ such that

$$\begin{aligned}\left(\sum_{\|w\|_\infty > \beta} |\widehat{Tf}(w)|^2\right)^{1/2} &\leq \left(\sum_{j=1}^k \sum_{(j+1)\beta \geq \|w\|_\infty > j\beta} |\widehat{Tf}(w)|^2\right)^{1/2} \leq \left(\sum_{j=1}^k (j+1)^d \beta^d C_{f,1} e^{-\frac{(j\beta)^{C_3}}{C_{f,3}}}\right)^{1/2} \\ &\leq C_{f,4,d}(\beta)^{d/2} e^{-\frac{\beta^{C_3}}{C_{f,5}}}.\end{aligned}$$

Let $\beta \in \mathbb{N}$, we decompose $f$ into the following form:

$$f(x) = a_0 + g(x) + R(x),$$

where

$$g(x) = \sum_{w \in \mathbb{Z}^d \cap [-\beta,\beta]^d \backslash \{0\}} a_w \exp(2\pi i w \cdot x),$$

and

$$R(x) = \sum_{w \notin \mathbb{Z}^d \cap [-\beta,\beta]^d} a_w \exp(2\pi i w \cdot x).$$

Observe that

$$\#\{w | w \in \mathbb{Z}^d \cap [-\beta,\beta]^d \backslash \{0\}\} = (2\beta+1)^d - 1,$$

and $\|R_2\|^2_{L^2([0,1]^d)} \leq C_{f,4,d}(\beta)^{d/2} e^{-\frac{\beta^{C_3}}{C_{f,5}}}$, Choosing $K = (2\beta+1)^d - 1$ and $\sigma_2 = 1/(2\ln L)$ in Theorem 2.6, we complete the proof. □

**Remark 3.2.** *Choosing $\sigma = 1/(2\ln L)$ and $\epsilon = \exp(-\frac{n^{C_*}}{dC_{f,*}})$ in Eq.(43), we know that the sample size is $L$ and the error is semi-exponential with respect to $L$ with probability at least $1 - 1/\ln L$, this result can be compared with Theorem 2 in [41] . Moreover, Eq.(45) represents the RMSE error after repeating $t$ times, which can also be compared with Corollary 3 in [41]. In [41], the upper bound of the convergence rate is $C_f M^{-C\log(M)/d}$ ($M$ denotes sample size), where $C_f$ is some constant depend on $f$ and $d$, the constant $C$ is an absolute constant satisfying $C < 0.21$. Our convergence order is significantly superior to the result in [41] in an asymptotic sense, although the constant in the exponential term of our error bounds is dependent on $f$.*

**Remark 3.3.** *From the perspective of sample complexity, our theorem reflects the **existence** of $M$ samples that can achieve a semi-exponential convergence rate. However, to achieve high precision with this method, it also involves approximating the integral $\int_0^y \exp(-1/(\xi(1-\xi))$. We may consider an alternative approach for future research, where for any sample size $M$, we utilize B-splines to construct a $k_M$ order smooth function $\psi_{k_M}$ as a substitute for the function $\psi$ defined in Eq. (47).*

We further deduce the subsequent theorem pertaining to the unit ball of the subspace of the Wiener algebra introduced in [16]:

$$\mathcal{A}_\kappa := \left\{ f \in C(\mathcal{T}^d) \middle| \|f\|_{\mathcal{A}} := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}(\boldsymbol{k})\kappa(\min_{j\in \operatorname{supp}(k)} |k_j|)| < \infty \right\},$$

where $\kappa(x)$ is a increasing function on $(0,+\infty)$ and $\operatorname{supp}(k) := \{j : 1 \leq j \leq d, k_j \neq 0\}$. For any $0 < \epsilon \leq 1/2$, there exits a positive number $N_{\kappa,\epsilon}$ such that $\kappa(N_{\kappa,\epsilon}) \geq 1/\epsilon$ .

**Note that:** In Theorem 3.4, We use $I(f)_{H_{N_j},L,r,z_j}$ instead of $I(f_{N,\eta_j})_{H_{N_j},L,r,z_j}$.

**Theorem 3.4.** *Let $f \in \mathcal{A}_\kappa$. Let $n > 2$, $0 < \sigma < 1/10$, $0 < \epsilon < \sigma/n$, $B = \lceil 4n/\sigma \rceil$, $r = B\sqrt{\log(1/\epsilon)}$, $L = \lceil r\sqrt{2\log(1/\epsilon)} \rceil$. Let $N \geq \max\{\lceil N_{\kappa,\epsilon}\rceil, 3L\}$ be a prime number, then it holds with probability at least $1 - \sigma - 1/10$ over the randomness of $H_N$, $z$ that*

$$\left| I(f)_{H_N,L,r,z} - \mathrm{INT}(f) \right| \leq \Theta\left(\frac{1}{n} + \epsilon\right). \tag{51}$$

*With median$\{I(f)_{H_{N_j},L,r,z_j}\}_{j=1}^t$ as in Theorem 2.6, it holds with probability at least $1 - (2\sqrt{\sigma + 1/10})^t$ that*

$$\left| median\{I(f)_{H_{N_j},L,r,z_j}\}_{j=1}^t - \mathrm{INT}(f) \right| \leq \Theta\left(\frac{1}{n} + \epsilon\right). \tag{52}$$

*Furthermore,*

$$\left(\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t}\left|median\{I(f)_{H_{N_j},L,r,z_j}\}_{j=1}^t-\mathrm{INT}(f)\right|^2\right)^{1/2}\le\Theta\left(\frac{1}{n}+\epsilon+(2\sqrt{\sigma+\frac{1}{10}})^t\right). \tag{53}$$

*Proof.* Let $f(x):=\sum a_w\exp(2\pi iw\cdot x)$, then $\sum|a_w|\le 1$. Let

$$\Xi_N=\{(t_1N,t_2N,\ldots,t_dN)|(t_1,t_2,\ldots,t_d)\in\mathbb{Z}^d\backslash\{0\}\}.$$

Since $N\ge\lceil N_{\kappa,\epsilon}\rceil$ and $f\in\mathcal{A}_\kappa$, we have

$$|\widetilde{f}^{(N)}(0)-a_0|\le\sum_{w\in\Xi_N}|a_w|\le(1/\kappa(N_{\kappa,\epsilon}))\sum_{w\in\Xi_N}|a_w|\kappa(\min_{j\in\operatorname{supp}(w)}|w_j|)\le\epsilon,$$

and

$$\sum_{w\in\mathbb{Z}_N^d}|\widetilde{f}^{(N)}(w)|\le 1,$$

where $\widetilde{f}^{(N)}(w)$ is defined in Eq. (2). Then, from Eq. (8), we have

$$\left|\sum_{|l|\le L}\widetilde{f}^{(N)}(0)G_{r,l}-a_0\right|\le\Theta(\epsilon).$$

We decompose $f$ as

$$f(z/N)=\widetilde{f}^{(N)}(0)+g(z/N), \tag{54}$$

It is obvious that $\widetilde{g}^{(N)}(0)=0$, let $b_w=\widetilde{g}^{(N)}(w)$.

Let $U_{H_N,B}:=\{w|H_N\cdot w\in\big[-\frac{N}{B},\frac{N}{B}\big](\mathrm{mod}N)\}$. By combining Eq. (8) and Eq. (32), we obtain

$$\Big|\sum_{|l|\le L}\mathbb{E}_{H_N}\big(\exp(2\pi iw\cdot(z-lH_N)/N\big|w\in U_{H_N,B}\big)G_{r,l}\mathbb{P}\{w\in U_{H_N,B}\}\le\Theta(\frac{1}{B}).$$

Using Eq. (7), we have

$$\Big|\sum_{|l|\le L}\mathbb{E}_{H_N}\big(\exp(2\pi iw\cdot(z-lH_N)/N\big|w\notin U_{H_N,B}\big)G_{r,l}\mathbb{P}\{w\notin U_{H_N,B}\}\le\Theta(\epsilon(1-\frac{1}{B})).$$

Therefore,

$$\mathbb{E}_{H_N}\left|\sum_{|l|\le L}g\left(\left\{\frac{z-lH_N}{N}\right\}\right)G_{r,l}\right|\le\sum_{w\in\mathbb{Z}_N^d\backslash\{0\}}\mathbb{E}_{H_N}|b_w|\left|\sum_{|l|\le L}\exp(2\pi iw\cdot(z-lH_N)/N)G_{r,l}\right|$$

$$=\sum_{w\in\mathbb{Z}_N^d\backslash\{0\}}|b_w|\Big|\sum_{|l|\le L}\Big(\mathbb{E}_{H_N}\big(\exp(2\pi iw\cdot(z-lH_N)/N\big|w\in U_{H_N,B}\big)G_{r,l}\mathbb{P}\{w\in U_{H_N,B}\}$$

$$+\mathbb{E}_{H_N}\big(\exp(2\pi iw\cdot(z-lH_N)/N\big|w\notin U_{H_N,B}\big)G_{r,l}\mathbb{P}\{w\notin U_{H_N,B}\}\}\Big)\Big|\le\Theta(\frac{1}{B}+\epsilon).$$

Recalling the condition $\epsilon < \sigma/n$ and and by selecting $H_N$ and $z$ uniformly from the sets $\mathbb{Z}_N^d \cap [1, N)^d$ and $z \in \mathbb{Z}_N^d$, respectively, we obtain by Markov's inequality that

$$\mathbb{P}\left\{ \left| \sum_{|l| \leq L} g\left( \left\{ \frac{z - lH_N}{N} \right\} \right) G_{r,l} \right| \leq \Theta\left( \frac{1}{n} \right) \right\} \geq 1 - \sigma. \tag{55}$$

Therefore, Eqs. (51) and (52) can be derived by summarizing Eqs. (54) and (55). Utilizing a similar method to the proof of Eq. (24), we are able to prove Eq. (53). □

**Remark 3.5.** *If we further assume that $f$ is Hölder continuous (see [3, 8]), a similar conclusion still holds. In this case, it suffices to replace $N_{\kappa,\epsilon}$ with a number related to Hölder continuity and $\epsilon$.*

**Remark 3.6.** *Form Eq.(5) and Eq. (53), we know that the sample size is $M := tL$ (the sample size $M$ is independent of the parameter $N$). Choosing $\sigma = 1/10$, $t = \log_{\sqrt{5}/2} n$ and $\epsilon = 1/n$ in Eq. (53), we know that the upper bound of the RMSE error is $\Theta((\log M)/M)$.*

**Remark 3.7.** *In [11] (see Theorem 3.1), it has been established that for Wiener-type functions with Fourier coefficients decaying sufficiently slowly, deterministic integral algorithms do not possess polynomial tractability. Our results indicate that this conclusion (Theorem 3.1 in [11] ) does not hold for random algorithms, provided that we have prior information regarding the decay of the Fourier coefficients of the functions.*

# 4 Integration in Weighted Sobolev Spaces

We will provide a theoretical analysis of our algorithm for integration in the weighted Sobolev space. Specifically, we consider the unit ball within this space, defined as

$$H_\gamma^{mix,s}(\mathcal{T}^d) = \left\{ f \in L^2(\mathcal{T}^d) \middle| \|f\|^2_{H^s_{mix}(\mathcal{T}^d)} := \sum_{w=(w_1,\ldots,w_d) \in \mathbb{Z}^d} (\Gamma_{\gamma,s}(w))^2 |\hat{f}(w)|^2 \leq 1 \right\},$$

where

$$\Gamma_{\gamma,s}(w) = (\gamma_{\operatorname{supp}(w)})^s \prod_{j=1}^d \max\{1, |w_j|^s\}$$

with the weight function $\gamma_{\operatorname{supp}(w)} \geq 1$ and $\operatorname{supp}(w) := \{j : 1 \leq j \leq d, w_j \neq 0\}$.

**Theorem 4.1.** *Let $f \in H_\gamma^{mix,s}(\mathcal{T}^d)$ with $s > 0$. Set $\alpha \geq 1$, $1/100 < \sigma < 1/20$, $n > 1$, $0 < \epsilon < \sigma/n$, $B = \lceil 4n/\sigma \rceil$, $r = B\sqrt{\log(1/\epsilon)}$, $L = \lceil r\sqrt{2\log(1/\epsilon)} \rceil$, and $N = \Theta(B^4/(\sigma\epsilon^4)^{1/d})$ to be a prime number. Under the conditions $L < N/3$, it holds with probability at least $1 - \sigma - 3/20$ over the randomness of $H_N$ and $z$ that*

$$\left| I(f_{N,\eta})_{H_N,L,r,z} - \mathrm{INT}(f) \right| \leq \Theta\left( C_{\alpha,s,\gamma,N} n^{-s/\alpha - 1/2} + \frac{1}{n^{s+1/2}} + \epsilon \right). \tag{56}$$

*It also holds with probability at least $1-(2\sqrt{\sigma+3/20})^t$ that*

$$\left|median\{I(f_{N,\eta_j})_{H_{N_j},L,r,z_j}\}_{j=1}^t - \mathrm{INT}(f)\right| \le \Theta(C_{\alpha,s,\gamma,N} n^{-s/\alpha-1/2} + \frac{1}{n^{s+1/2}} + \epsilon). \quad (57)$$

*Furthermore,*

$$\begin{aligned}&\left(\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t}\left|median\{I(f_{N,\eta})_{H_{N_j},L,r,z_j}\}_{j=1}^t - \mathrm{INT}(f)\right|^2\right)^{1/2} \\ &\le \Theta\left(C_{\alpha,s,\gamma} n^{-s/\alpha-1/2} + \frac{1}{n^{s+1/2}} + \epsilon + (2\sqrt{\sigma+3/20})^t\right),\end{aligned} \quad (58)$$

*where $C_{\alpha,s,\gamma,N} = (\sum_{w\in Q_{N/2}\setminus\{0\}} \frac{1}{(\Gamma_{\gamma,s}(w))^{\alpha/s}})^{1/2+s/\alpha}$, $Q_{N/2} = \mathbb{Z}_N^d \cap (-N/2, N/2)^d$.*

**Remark 4.2.** *Form Eq.(5) and Eq. (58), we know that the sample size is $M := tL$. Choosing $\sigma = 1/20$, $t = \log_{\sqrt{5}/2}(n^{s+1/2})$, and $\epsilon = 1/n^{s+1/2}$ in Eq. (58), we know the upper bound of the RMSE error is*

$$C'(C'' + C_{\alpha,s,\gamma,N})(3/2+s)^{s+3/2}(\log M)(\log M/M)^{s+1/2}, \quad (59)$$

*where $C'$, $C''$ is some absolute constant. When $\alpha = 1$, $\gamma_{\mathrm{supp}\,(w)} = 1$ then $C_{\alpha,s,\gamma,N} \le (\Theta(1)\log M))^d$. When $\alpha > 1$ and $\gamma_{\mathrm{supp}\,(w)} = \prod_{j\in \mathrm{supp}\,(w)}(1/\gamma_j), \gamma_j \le 1$, the RMSE error 59 is very close to the RMSE error in [31] (with no clear superiority discernible between the two). Both algorithms are polynomial tractable, provided that we treat $\sum_{j=1}^d \gamma_j := C_\gamma$ and $\zeta(\alpha)$ as the constants (as done in [31]), where $\zeta$ denotes the Riemann zeta function.*

**Remark 4.3.** *Very recently, [19] provided a nearly optimal algorithm for the case where the order $s > 1/2$ and $\sum_{j=1}^d \gamma_j := C_\gamma < \infty$. Their algorithm also does not require prior information on the weights, additionally, does not require prior information on the order $s$. In contrast, our algorithm needs to set a minimum accuracy $\epsilon$, meaning that to achieve the nearly optimal convergence order, we need to know an upper bound for $s$ (although the exact value of $s$ is not required). On the other hand, our algorithm is applicable to the scenario where the order $0 < s \le 1/2$, implying that it is suitable for discontinuous and non-periodic functions. As mentioned previously, these two works are independent of each other and different in terms of algorithms.*

*Proof.* Our proof draws inspiration from [31]. For $\alpha \ge 1$, Define

$$V_{d,\alpha,N} := \sum_{w\in Q_{N/2}\setminus\{0\}} \frac{1}{(\Gamma_{\gamma,s}(w))^{\alpha/s}}.$$

According to Lemma 2.4, for $N/3 \ge B \ge 4n/\sigma$,

$$\mathbb{E}_{H_N} \sum_{w\in Q_{N/2}\setminus\{0\}} \mathbb{I}_{w\in U_{H_N,B}} \frac{1}{(\Gamma_{\gamma,s}(w))^{\alpha/s}} \le \frac{\sigma V_{d,\alpha}}{n},$$

where $U_{H_N,B} := \{w | H_N \cdot w \in \left[-\frac{N}{B}, \frac{N}{B}\right] \pmod{N}\}$. Employing Markov's inequality, we derive

$$\sum_{w\in Q_{N/2}\setminus\{0\}} \mathbb{I}_{w\in U_{H_N,B}} \frac{1}{(\Gamma_{\gamma,s}(w))^{\alpha/s}} \le \frac{\sigma V_{d,\alpha,N}}{n\sigma} = \frac{V_{d,\alpha,N}}{n},$$

with probability at least $1-\sigma$ over the randomness of $H_N$. Consequently, with probability at least $1-\sigma$,

$$w \notin U_{H_N,B} \quad \text{for} \quad \text{all} \quad (\Gamma_{\gamma,s}(w))^{\frac{1}{s}} < \left(\frac{n}{V_{d,\alpha,N}}\right)^{\frac{1}{\alpha}}. \tag{60}$$

Let $U_1 := \left\{ w \in \mathbb{Z}^d \cap [-B,B]^d \backslash \{0\} \middle| (\Gamma_{\gamma,s}(w))^{\frac{1}{s}} < \left(\frac{n}{V_{d,\alpha,N}}\right)^{\frac{1}{\alpha}} \right\}$. For any $f \in H^s_{mix}(\mathcal{T}^d)$, we decompose $f$ as

$$f(x) = a_0 + g(x) + R(x),$$

where $a_0 = \mathrm{INT}(f)$,

$$g(x) = \sum_{w \in U_1} a_w \exp(2\pi i w \cdot x), \text{ and } R(x) = \sum_{w \in \mathbb{Z}^d_N \backslash (U_1 \cup \{0\})} b_w \exp(2\pi i w \cdot x).$$

Since $\gamma_{supp(w)} \in [1,\infty)$,

$$\|R\|_{L^2([0,1]^d)} \le \frac{1}{B^s} + \left(\frac{V_{d,\alpha,N}}{n}\right)^{2s/\alpha}.$$

For $z \in \mathbb{Z}^d_N$,

$$f_{N,\eta}(z/N) = a_0 + g_{N,\eta}(z/N) + R_{N,\eta}(z/N). \tag{61}$$

Observe that $E_\eta \frac{1}{N^d} \sum_{w \in \mathbb{Z}^d_N} |R_{N,\eta}(z/N)|^2 = \|R\|^2_{L^2([0,1]^d)}$. By Markov's inequality,

$$\mathbb{E}_z |R_{N,\eta}(z/N)|^2 = \frac{1}{N^d} \sum_{w \in \mathbb{Z}^d_N} |R_{N,\eta}(z/N)|^2 \le 20 \|R\|^2_{L^2([0,1]^d)} \le 20 \left( \frac{1}{B^s} + \left(\frac{V_{d,\alpha,N}}{n}\right)^{2s/\alpha} \right), \tag{62}$$

with probability at least $1 - 1/20$.

$f_{N,\eta}(z/N)$ can also be expressed in the following form.

$$f_{N,\eta}(z/N) = \widetilde{f}^{(N)}_{N,\eta}(0) + g(z/N) + R^*(z/N), \quad z \in \mathbb{Z}^d_N, \tag{63}$$

where

$$R^*(z/N) = R_{N,\eta}(z/N) + g_{N,\eta}(z/N) - g(z/N) + a_0 - \widetilde{f}^{(N)}_{N,\eta}(0).$$

Since $a_0 = \mathrm{INT}(f)$, the integral error is comprised of the following three components:

$$\left| \sum_{|l| \le L} \widetilde{f}^{(N)}_{N,\eta}(0) G_{r,l} - a_0 \right|, \quad \left| \sum_{|l| \le L} g\left( \left\{ \frac{z - lH_N}{N} \right\} \right) G_{r,l} \right|, \quad \left| \sum_{|l| \le L} R^*\left( \left\{ \frac{z - lH_N}{N} \right\} \right) G_{r,l} \right|.$$

Similar to the proof of Lemma 2.5, it holds that

$$|\widetilde{f}^{(N)}_{N,\eta}(0) - a_0| \le \epsilon \tag{64}$$

with probability at least $1 - 1/(N^d \epsilon^2)$. Using Eq.(8), we have

$$\left| \sum_{|l| \le L} \widetilde{f}^{(N)}_{N,\eta}(0) G_{r,l} - a_0 \right| \le \Theta(\epsilon) \tag{65}$$

Using Eq. (60), Lemma 2.3, and Eqs. (33, 34), we obtain

$$\mathbb{E}_z\Big|\sum_{|l|\le L} g\Big(\Big\{\frac{z-lH_N}{N}\Big\}\Big)G_{r,l}\Big|^2 \le \epsilon^2\big(\sum|a_w|^2\big) \le \epsilon^2,$$

which holds with probability at least $1-\sigma$ over the randomness of $H_N$. By selecting $z \sim \text{unif } \mathbb{Z}_N^d$, and applying Markov's inequality, we derive

$$\Big|\sum_{|l|\le L} g\Big(\Big\{\frac{z-lH_N}{N}\Big\}\Big)G_{r,l}\Big|^2 \le 40\epsilon^2 \tag{66}$$

with probability at least $1-\sigma-1/40$ over the randomness of $H_N$ and $z$.

Lastly, we estimate $\Big|\sum_{|l|\le L} R^*\Big(\Big\{\frac{z-lH_N}{N}\Big\}\Big)G_{r,l}\Big|$. Similar to the proof of Lemma 2.5, for each $w\in U_1$, $\mathbb{E}_\eta\widetilde{g}^{(N)}_{N,\eta}(w)=a_w$ and $\mathbb{E}_\eta|\widetilde{g}^{(N)}_{N,\eta}(w)-a_w|^2\le 1/N^d$. Thus, by Chebyshev's inequality,

$$|\widetilde{g}^{(N)}_{N,\eta}(w)-a_w|\le \epsilon/(2B+1)^d \quad \text{for} \quad \text{every} \quad w\in U_1$$

with probability at least $1-(2B+1)^{3d}/(N^d\epsilon^2)$. On the other hand, since $\mathbb{E}_\eta\mathbb{E}_z|g_{N,\eta}(z/N)|^2 = \|g(x)\|^2_{L^2([0,1]^d)}$ and

$$|g_{N,\eta}(z/N)|^2 = |\sum_{w\in U_1} a_\xi \exp(2\pi i w\cdot\eta_z)\exp(2\pi i w\cdot z/N)|^2 \le (2B+1)^{2d},$$

we have

$$\mathbb{E}_\eta|\mathbb{E}_z|g_{N,\eta}(z/N)|^2 - \|g(x)\|^2_{L^2([0,1]^d)}|^2 \le (2B+1)^{4d}/N^d \le (3B)^{4d}/N^d.$$

Again, by Chebyshev's inequality, it holds with probability at least $1-(3B)^{4d}/(N^d\epsilon^4)$ that

$$\Big|\sum_{w\in\mathbb{Z}_M^d}|\widetilde{g}^{(N)}_{N,\eta}(w)|^2 - \sum_{w\in U_1}|a_w|^2\Big| = |\mathbb{E}_z|g_{N,\eta}(z/N)|^2 - \|g(x)\|^2_{L^2([0,1]^d)}| \le \epsilon^2.$$

Choosing $N=\Theta((3B)^4/(\sigma\epsilon^4)^{1/d})$, we have with probability at least $1-\sigma$ that

$$\begin{aligned} \mathbb{E}_z|g_{N,\eta}(z/N)-g(z/N)|^2 &\le \sum_{w\in U_1}|\widetilde{g}^{(N)}_{N,\eta}(w)-a_w|^2 + \sum_{w\in\mathbb{Z}_N^d\setminus U_1}|\widetilde{g}^{(N)}_{N,\eta}(w)|^2 \\ \le \epsilon^2 + (\sum_{w\in\mathbb{Z}_N^d}|\widetilde{g}^{(N)}_{N,\eta}(w)|^2 - \sum_{w\in U_1}|a_w|^2) &+ (\sum_{w\in U_1}|a_w|^2 - \sum_{w\in U_1}|\widetilde{g}^{(N)}_{N,\eta}(w)|^2) \le \Theta(\epsilon^2). \end{aligned} \tag{67}$$

Using Eqs. (62),(64), (67), it holds that

$$\mathbb{E}_z(R^*(z/N))^2 = \frac{1}{N^d}\sum_{z\in\mathbb{Z}_N^d}(R^*(z/N))^2 \le 40\Big(\frac{1}{B^{2s}} + \Big(\frac{V_{d,\alpha,N}}{n}\Big)^{2s/\alpha}\Big) + \Theta(\epsilon^2)$$

with probability at least $1 - 2\delta - 1/20$ over the randomness of $\eta$. Let

$$R^*(z/N) := \sum_{z \in \mathbb{Z}_N^d} q_w \exp(2\pi i w \cdot z/N),$$

thus

$$\sum |q_w|^2 \leq 40\left(\frac{1}{B^{2s}} + \left(\frac{V_{d,\alpha,N}}{n}\right)^{2s/\alpha}\right) + \Theta(\epsilon^2),$$

with probability at least $1 - 2\delta - 1/20$.

Analogously, following the proof of Eq. (35), we have

$$\begin{aligned}\mathbb{E}_{H_N}\mathbb{E}_z\left|\sum_{|l|\leq L} R^*\left(\left\{\frac{z - lH_N}{N}\right\}\right)G_{r,l}\right|^2 &\leq \Theta\left(\frac{\sum |q_w|^2}{B}\right)\\ &\leq \Theta\left(\frac{1}{B}\left(40\left(\frac{1}{B^{2s}} + \left(\frac{V_{d,\alpha,N}}{n}\right)^{2s/\alpha}\right) + \Theta(\epsilon^2)\right)\right).\end{aligned} \tag{68}$$

Recalling the conditions $\epsilon < \sigma/n$ and $B = \lceil 4n/\sigma \rceil$. we randomly select $H_N \sim$ unif $\mathbb{Z}_N^d \cap [1, N)^d$ and $z \sim$ unif $\mathbb{Z}_N^d$, respectively. By Markov's inequality, we then obtain with probability at least $(1 - \sigma)(1 - 2\sigma - 1/20)$ that

$$\left|\sum_{|l|\leq L} R^*\left(\left\{\frac{z - lH_N}{N}\right\}\right)G_{r,l}\right|^2 \leq \Theta\left(\left(\frac{V_{d,\alpha,N}}{n}\right)^{1+2s/\alpha} + \frac{1}{n^{2s+1}} + \frac{\epsilon^2}{n}\right). \tag{69}$$

Combining Eqs. (65), (66) and (69), the total failure probability is at most $5\sigma + 1/8$. Consequently, Eqs. (56) and (57) are proven. Following a similar approach to the proof of Eq. (24), we can successfully demonstrate the validity of Eq. (58). □

# 5 Numerical Experiments

In this section, we employ the numerical examples from existing literature to demonstrate that our algorithm is implementable and capable of achieving a convergence order comparable to those reported in the literature. It is crucial to point out that while the objective of this paper is to propose an algorithm that **theoretically** enjoys high convergence order, this does not imply that our algorithm outperforms existing algorithms in terms of specific numerical examples. Further analysis on this aspect is conducted in the "Future work" section.

For our numerical experiments, we shall select a differentiable function $f_1$, a non-differentiable function $f_2$ and a discontinuous function $f_3$. These three functions are given by

$$f_1(x) = \prod_{j=1}^{d}\left[1 + \frac{1}{j^4}B_4(x_j)\right],$$

$$f_2(x) = \prod_{j=1}^{d}\left[1 + \frac{|4x_j - 2| - 1}{j^2}\right] - 1,$$

$$f_3(x) = \mathbb{I}_{\left\{\sum_{i=1}^d x_i \geq d/2\right\}}(x),$$

where $B_4$ is the Bernoulli polynomial of degree 4.

The examples $f_1$ and $f_2$ were suggested in [10] to validate the nearly optimal performance of random component-by-component algorithm for integration in weighted Sobolev space. Meanwhile, the example $f_3$ was utilized in [24, 25] to discuss the efficacy of integration algorithms for discontinuous functions. Although $f_3$ can still be considered as a function in weighted Sobolev space, it falls outside the scope of the algorithm proposed in [10] due to its order being less than 1/2.

We define the squared error as

$$\text{error} := \left|median\{I(f_{N,\eta_j})_{H_{N_j},L,r,z_j}\}_{j=1}^t - \text{INT}(f)\right|^2.$$

Here, the number of samples is denoted by $M := 2L+1$ (referring to Eq. (5)). Consistent with [20], we exclude the repetition times $t$ from the calculation of the total sample size. To ensure stability in our results, we uniformly select a large number $t = 63$ of repetitions. In fact, if our goal is to achieve an MSE bound similar to Eq. (58), we would only require selecting $t = \Theta((s+1/2)\log L))$. For the functions $f_1$ $f_2$, we set the dimension as $d = 20$, and the value of $N$ as 5600748293801. For the functions $f_3$, we set the dimension as $d = 500$, and the value of $N$ as 5600748293801, and we employ the tent transformation to impart periodicity to it, resulting in the following form

$$f_3^*(x) := f_3(\varphi(x)) = f_3((1 - |2x_1 - 1|, 1 - |2x_2 - 1|, \ldots, 1 - |2x_d - 1|)).$$

The numerical results for $f_1$, achieved through various values of $M$ and $r$, are exhibited in Fig 1 , clearly highlighting that the convergence order of our method surpasses $1/M^5$. Similarly, the numerical results for $f_2$, displayed in Fig 2, indicate a convergence order approaching $1/M^3$. Notably, for both test functions $f_1$ and $f_2$, our convergence rate is comparable to that reported in [10], achieved without any prior knowledge of the weights.

The numerical results for $f_3$, illustrated in Fig 3, reveal that in the 500-dimensional scenario, the convergence order of our algorithm approaches $M^{-1.5}$, marking a substantial improvement over the standard Monte Carlo method, which is of order $1/M$. This superiority remains evident even taking into account the number of repetitions. In fact, to achieve an MSE bound as demonstrated in Eq. (58), it suffices to set $t$ as $\Theta((s + 1/2)\log M)$. As $M$ increases, the impact of $\Theta((s + 1/2)\log M)$ becomes progressively negligible. In contrast, the algorithms presented in [25, 24] tends to converge towards the standard Monte Carlo rate with increasing dimensionality, particularly in the 8-dimensional scenario, their convergence rate was documented as $1/M^{1.054}$ [25], which served to illustrate their theoretical upper bound of order $1/M^{1+1/d}$ for the scrambled net method.

**Remark 5.1.** *We have set the window function to be* $G_{r,l} = \frac{1}{r\sqrt{2\pi}}\exp(-\frac{l^2}{2r^2})$ *instead of* $G_{L,r,l} = \sum_{l+kN\leq L} \frac{1}{r\sqrt{2\pi}}\exp(-\frac{(l+kN)^2}{2r^2})$ *which originally defined in the initial version of the paper [5]. When* $N > 3L$*, the two are equivalent, thus we have not recalculated our numerical examples in this version.*

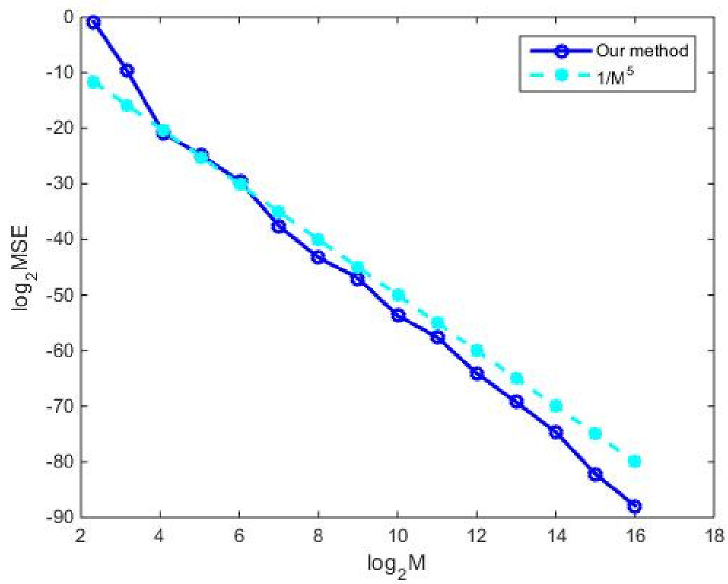


Figure 1: Convergence behavior of our method for $f_1$ with $r = 0.228 \times 2^k$ ($k = 0, 1, \ldots, 14$) and $L = 2^k$ ($k = 1, 2, \ldots, 15$). The mean squared errors are computed based on 30 runs.

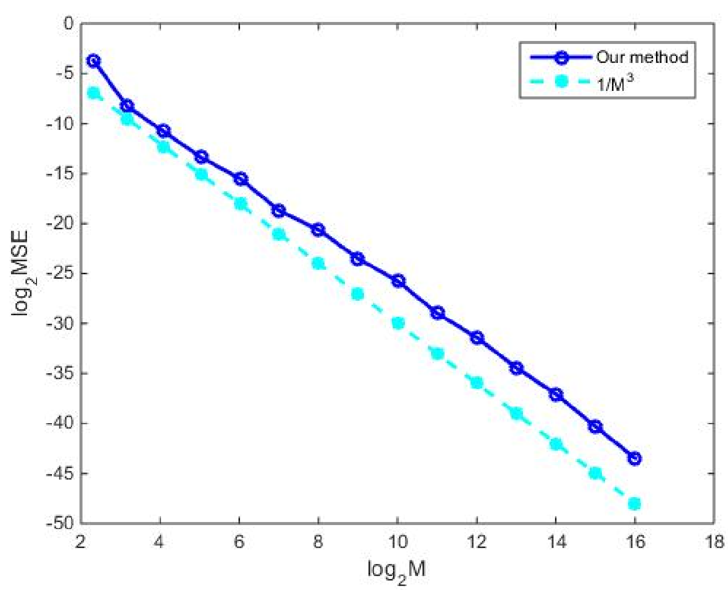


Figure 2: Convergence behavior of our method for $f_2$ with $r = 0.32 \times 2^k$ ($k = 0, 1, \ldots, 14$) and $L = 2^k$ ($k = 1, 2, \ldots, 15$). The mean squared errors are computed based on 30 runs.

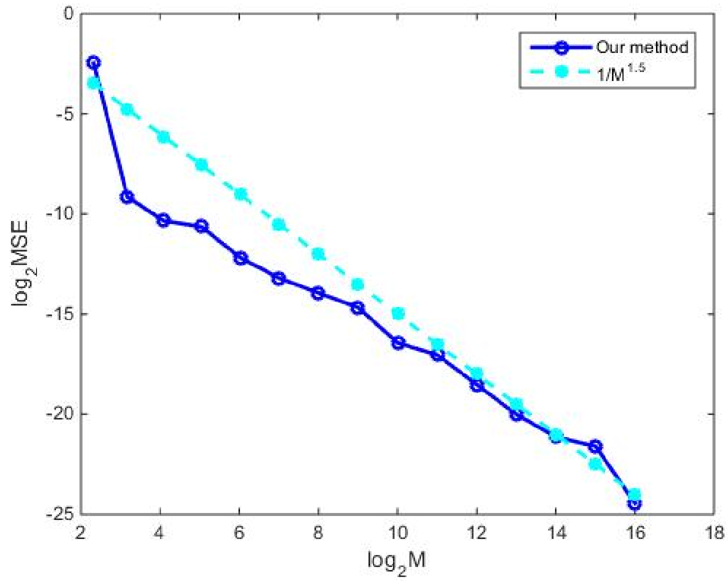


Figure 3: Convergence behavior of our method for $f_3$ with $r = 0.42 \times 2^k (k = 0, 1, \ldots, 14)$ and $L = 2^k$ $(k = 1, 2, \ldots, 15)$. The mean squared errors are computed based on 30 runs.

# 6 Future Work

We have theoretically developed a nearly optimal integration algorithm for periodic isotropic Sobolev space and weighted Sobolev space, which boasts polynomial tractability. However, the RMSE upper bounds for these two spaces involve the factors $\log^{\frac{1}{2}+\frac{s}{d}}(M)$ and $\log^{\frac{1}{2}+s}(M)$ ($M$ denotes sample size), respectively, thus hampering the algorithm's practical performance. We believe that the integration algorithm combining the median trick with the following random lattice rule can achieve the nearly optimal bounds under the addition of some mild conditions.

$$I(f) := \frac{1}{M} \sum_{0 \le l < M} f_{\eta,M}\left(\left\{\frac{z - lH_M}{M}\right\}\right),$$

where $M$ is a prime, $H_M$ and $z$ are drawn from the uniform distribution over $\mathbb{Z}^d \cap [1, M)^d$ and $\mathbb{Z}^d \cap [0, M)^d$, respectively, and the definition of $f_{\eta,M}$ is given in Eq.(12). Furthermore, this algorithm has the potential to reduce the logarithmic factors in the upper bound. We leave the exploration of this algorithm for future research.

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